Exercise 20

In Exercises 17–24, find the unknown if the solution of each equation is given:

$$\text{If } u(x) = e^{-x^2} \text{ is a solution of } u(x) = 1 - \alpha \int_0^x tu(t)\,dt, \text{ find } \alpha$$

Solution

Substitute the solution into both sides of the equation.

$$e^{-x^2} = 1 - \alpha \int_0^x te^{-t^2}\,dt$$

Make a substitution to evaluate the integral.

$$\begin{aligned} v &= t^2 \\ dv &= 2t\,dt \quad \rightarrow \quad \frac{dv}{2} = t\,dt \end{aligned}$$

The equation becomes

$$\begin{aligned} e^{-x^2} &= 1 - \alpha \int_0^{x^2} e^{-v} \left(\frac{dv}{2}\right) \\ &= 1 - \frac{\alpha}{2} \int_0^{x^2} e^{-v}\,dv \\ &= 1 - \frac{\alpha}{2}(-e^{-v})\Big|_0^{x^2} \\ &= 1 - \frac{\alpha}{2}(-e^{-x^2} + 1). \end{aligned}$$

Subtract 1 from both sides.

$$e^{-x^2} - 1 = -\frac{\alpha}{2}(-e^{-x^2} + 1)$$

Multiply both sides by -2.

$$2(-e^{-x^2} + 1) = \alpha(-e^{-x^2} + 1)$$

Therefore,

$$\alpha = 2.$$